Table 1 Collapse loads

| $R/h = 100$ | | | |
|---|---|---|---|
| Cutout | | Critical load $\bar{P}$, with | |
| Type | Size, $a/R$ | Unif. load | Unif. displacement |
| Circ. | 0.6 | 0.236 | 0.510 |
| Circ. | 1.2 | 0.149 | 0.376 |
| Circ. | 2.4 | 0.035 | 0.303 |
| Square | 1.2 | 0.110 | 0.378 |

carry more load than those with square cutouts. A few cases were also analyzed with this loading condition, and the results are shown in Table 1. It can be seen that for this condition the critical loads are lower, especially for shells with square cutouts.

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# Shooting Method for Solution of Boundary-Layer Flows with Massive Blowing


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### Introduction

IN an earlier publication, Nachtsheim and Green[1] (1971) presented a numerical method to solve the boundary-layer equations with massive blowing. The method is based on bidirectional shooting starting from the dividing streamline. Recently, Liu and Nachtsheim[2] proved that, in the case of massive blowing, the parasitic eigenvalues associated with the boundary-layer equations equal the stream function with opposite sign and therefore are positive. The nonexistence of the negative parasitic eigenvalue guarantees that the backward shooting technique employed in Ref. 1 is stable. However, there is one drawback in Ref. 1, i.e., one cannot specify the blowing rate $f_w$ and the wall temperature $s_w$ a priori. This Note will present a modified shooting method to solve boundary-layer equations with massive blowing for specified $f_w$ and $s_w$.


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### Method Description

For boundary-layer flows with blowing, the stream function $f$ is negative near the wall and positive at the edge and there is a dividing streamline where the oncoming flow is stopped by the injected flow and $f(\eta_0) = 0$. For massive blowing, the location of the dividing streamline increases with blowing rate and becomes very large. In the inner region, $\eta < \eta_0$, the parasitic eigenvalues of the equations are positive and large,[2] and it is wise not to integrate the equations from the wall (unstable direction), but instead one should integrate backward from the dividing streamline (stable direction) as in Ref. 1. This is the key point for the stable numerical integration of the massively blown boundary-layer equations.

The modification of the shooting method of Ref. 1 is illustrated as follows. Consider the Cohen-Reshotko equations

$$f''' + ff'' + \beta(1+s-f'^2) = 0 \quad (1)$$

$$s'' + fs' = 0 \quad (2)$$

$$f(0) = -C, \quad f'(0) = 0, \quad s(0) = s_w \quad (3)$$

$$f'(\infty) = 1, \quad s(\infty) = 0 \quad (4)$$

This is the same set of equations considered in Ref. 1. Introduce a new independent variable

$$\zeta = (\eta - \eta_0)/\eta_0 \quad (5)$$

so that the wall is always located at $\zeta = -1$ and the dividing streamline is always located at $\zeta = 0$. This transformation avoids the difficulty of imposing wall conditions when $\eta_w$ (in the notations of Ref. 1) is unknown a priori. The same idea has been applied successfully by Liu and Libby[3] to solve the flame sheet model for stagnation point flows. With this transformation, the location of the dividing streamline $\eta_0$ will appear as an unknown parameter in the conservation Eqs. (1) and (2). Moreover, to avoid confusion, new dependent variables are defined as follows:

$$F(\zeta) = f(\eta), \quad g(\zeta) = s(\eta) \quad (6)$$

Substituting Eqs. (5) and (6) into Eqs. (1) to (4), we obtain the following:

$$F''' + \eta_0 FF'' + \beta[\eta_0{}^3(1+g) - \eta_0 F'^2] = 0 \quad (7)$$

$$g'' + \eta_0 Fg' = 0 \quad (8)$$

$$F(-1) = -C, \quad F'(-1) = 0, \quad g(-1) = s_w \quad (9)$$

$$F'(\infty) = \eta_0, \quad g(\infty) = 0 \quad (10)$$

$$F(0) = 0 \quad (11)$$

In this new set of equations, $\eta_0$ appears as a parameter that must be determined in the course of solution. We note that Eqs. (7–11) are a three-point, boundary-value problem; the additional boundary condition (11) is just the definition of the dividing streamline. Next, we generalize the automated procedure of Nachtsheim and Swigert[4] for satisfying the asymptotic boundary conditions at the outer edge of the boundary layer. Let

$$x_1 = F'(0), \quad x_2 = F''(0), \quad x_3 = g(0), \quad x_4 = g'(0) \quad (12)$$

where $x_i$, $i = 1, 2, 3, 4$, and $\eta_0$ must be chosen so that conditions (9) and (10) are satisfied. Differentiating Eqs. (7) to (8) with respect to $x_i$ and $\eta_0$, we obtain the following perturbation equations:

$$F_{x_i}''' + \eta_0[FF_{x_i}'' + F''F_{x_i} + \beta(\eta_0{}^2 g_{x_i} - 2F'F_{x_i})] = 0 \quad (13)$$

$$g_{x_i}'' + \eta_0[Fg_{x_i}' + g'F_{x_i}] = 0 \quad (14)$$

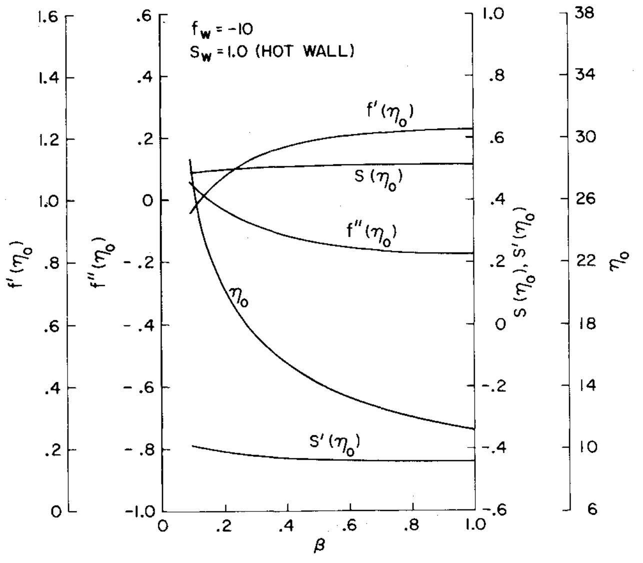


Fig. 1 Variation of velocity, shear, enthalpy, enthalpy gradient, and location of dividing streamline with $\beta$.

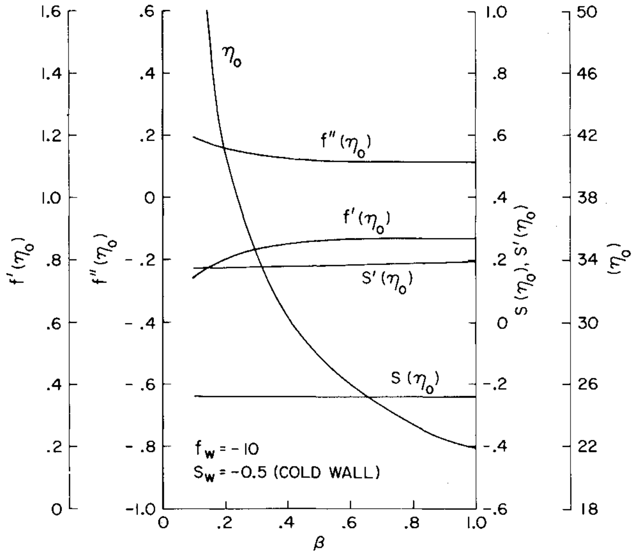


Fig. 2 Variation of velocity, shear, enthalpy, enthalpy gradient, and location of dividing streamline with $\beta$.

$$F_{\eta_0}''' + \eta_0[FF_{\eta_0}'' + F''F_{\eta_0} + \beta(\eta_0^2 g_{\eta_0} - 2F'F_{\eta_0}')] = -FF'' - \beta[3\eta_0^2(1+g) - F'^2] \quad (15)$$

$$g_{\eta_0}'' + \eta_0[Fg_{\eta_0}' + g'F_{\eta_0}] = -Fg' \quad (16)$$

The initial conditions to Eqs. (13) to (14) are zero except

$$F_{x_1}'(0) = F_{x_2}''(0) = g_{x_3}(0) = g_{x_4}'(0) = 1 \quad (17)$$

The initial conditions to Eqs. (15) to (16) are all zero. Equations (7) and (8) and (13–16) are then integrated with proper initial conditions (12) and (17). The exact solution of $x_i$, $i = 1, 2, 3, 4$, and $\eta_0$ have to be solved iteratively. In each iteration, we consider the discrepancies $\delta_i$, $i = 1, \ldots, 7$. For $\zeta \to \infty$

$$\delta_1 = (F' - \eta_0) + \sum_{i=1}^{4} F_{x_i}'\Delta x_i + F_{\eta_0}'\Delta\eta_0 \quad (18)$$

$$\delta_2 = F'' + \sum_{i=1}^{4} F_{x_i}''\Delta x_i + F_{\eta_0}''\Delta\eta_0 \quad (19)$$

$$\delta_3 = g + \sum_{i=1}^{4} g_{x_i}\Delta x_i + g_{\eta_0}\Delta\eta_0 \quad (20)$$

$$\delta_4 = g' + \sum_{i=1}^{4} g_{x_i}'\Delta x_i + g_{\eta_0}'\Delta\eta_0 \quad (21)$$

For $\zeta = -1$

$$\delta_5 = F + C + \sum_{i=1}^{4} F_{x_i}\Delta x_i + F_{\eta_0}\Delta\eta_0 \quad (22)$$

$$\delta_6 = F' + \sum_{i=1}^{4} F_{x_i}'\Delta x_i + F_{\eta_0}'\Delta\eta_0 \quad (23)$$

$$\delta_7 = g - s_w + \sum_{i=1}^{4} g_{x_i}\Delta x_i + g_{\eta_0}\Delta\eta_0 \quad (24)$$

We want to find $\Delta x_i$, $i = 1, 2, 3, 4$, and $\Delta\eta_0$ so that the sum of the square of the discrepancies

$$E = \sum_{i=1}^{7} \delta_i^2 \quad (25)$$

is as small as possible. Differentiating Eq. (25) with respect to $\Delta x_i$, $i = 1, 2, 3, 4$, and $\Delta\eta_0$, we obtain a system of five simultaneous linear equations in $\Delta x_i$, $i = 1, 2, 3, 4$, and $\Delta\eta_0$. After solving the resulting systems of equations, $x_i$, $i = 1, 2, 3, 4$, and $\eta_0$ will be updated in the Newton-Raphson procedure until the error, $E$, in Eq. (25) is less than a prescribed value.

We note that the procedure in Ref. 4 has been proved to be equivalent to requiring that the vorticity has the right asymptotic behavior and, therefore, the solutions thus obtained are unique.[5]

### Numerical Results

The method previously described was programed in FORTRAN IV on the IBM 360-67 located at Ames Research Center. The procedure was used to obtain solutions with large values of $C$, the blowing parameter, for various values of $\beta$, the pressure gradient parameter, and $s_w$, the wall temperature parameter. The boundary-layer and perturbation differential equations were integrated with Runge-Kutta-Gill subroutines.

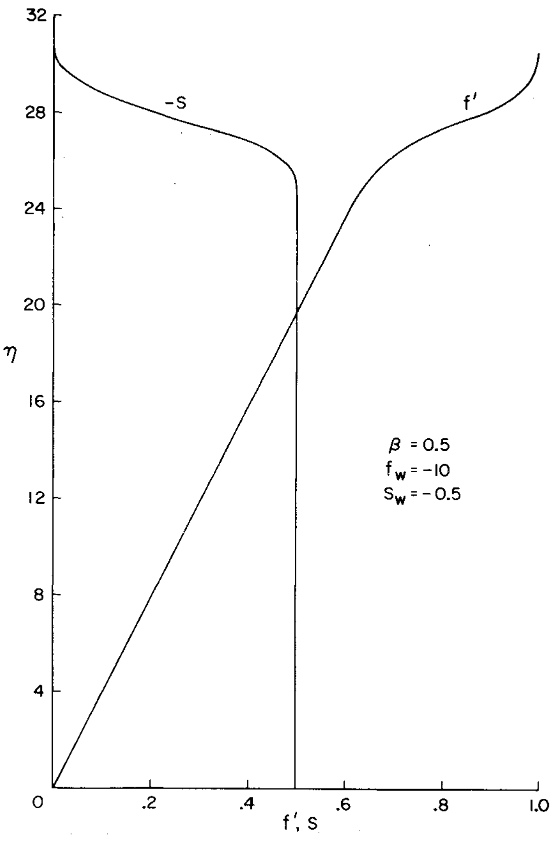


Fig. 3 Velocity and temperature profiles.

Figure 1 presents the variations of the velocity, shear, enthalpy, enthalpy gradient, and the location of dividing streamline with pressure gradient parameter $\beta$ for the case of a hot wall with $f_w = -10$. We note that, in this case, the velocity has an overshoot. The results show that the maximum overshoot is near the dividing streamline. Figure 2 presents the variations of the same quantities as in Fig. 1 with $\beta$, for a cold wall with $f_w = -10$. In both cases, the location of the dividing streamline $\eta_0$ increases as $\beta$ decreases. As has been proved by Kassoy,[6] to lowest order, $\eta_0$ is proportinal to $\beta^{-1/2}$. This has been confirmed by the present calculations. Figure 3 presents the velocity and temperature profiles for a cold wall, $s_w = -0.5$, with $f_w = -10$, at a stagnation point $\beta = 0.5$. The agreement with Libby's[7] analytic result for inner inviscid solutions is excellent.

### Conclusions

The calculations performed using the bidirectional shooting method indicate that it is possible to integrate the boundary-layer equations under conditions of massive blowing. Unlike the conventional shooting method, which is unstable when the blowing rate increases, the proposed method avoids the unstable direction, instead integrates from the dividing streamline in both directions, i.e., toward the boundary-layer edge and toward the wall. The method proposed to solve the three-point boundary conditions is capable of solving complex boundary-layer problems involving mass and energy balance on the surface.

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# Derivation of Aerodynamic Kernel Functions

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RECENTLY there has been considerable interest in improving upon the potential flow aerodynamic model for determining the pressure distribution on lifting surfaces. In the present Note the method of Fourier Transforms is used to determine the Kernel Function which relates the pressure to the (prescribed) downwash within the framework of the shear flow model of Ref. 1. Such a model is intended to allow for the effects of an aerodynamic boundary layer which is, of course, neglected in the potential flow model. For simplicity we consider incompressible, steady flow. However the added difficulties associated with compressibility and nonsteadiness are largely computational rather than conceptual. We first illustrate the proposed method by rederiving known results from potential flow theory.

Fleeter[2] has recently used Fourier Transform methods to determine the Kernel Function for potential flow past a two-dimensional cascade of airfoils. Miles[3] has previously used such methods for the wind-tunnel correction problem. Several investigators, including Miles, Yates, and the two present authors have considered nonlifting "thickness" problems.

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### Potential Flow

*Two dimensions*

The pressure satisfies Laplace's equation (in $x$ and $z$)

$$\nabla^2 p = 0 \tag{1}$$

as well as boundary conditions

$$\text{finiteness at infinity} \tag{2}$$

$$\partial p/\partial z = -\rho_\infty U_\infty \partial w/\partial x \qquad \text{on wing} \tag{3}$$

$$p = 0 \qquad \text{off wing} \tag{4}$$

$w$ is the downwash (for steady flow, it is simply the slope times the freestream velocity) of the airfoil. Denote by an * the Fourier Transform with respect to streamwise coordinate $x$, e.g.,

$$p^* \equiv \int_{-\infty}^{\infty} p e^{-i\alpha x} dx \tag{5}$$

Transforming Eq. (1) and solving, using Eqs. (2) and (3), one obtains (on $z = 0$, the plane of the airfoil)

$$p^*/\rho_\infty U_\infty^2 = (i\alpha/|\alpha|) w^*/U_\infty \tag{6}$$

If one knew the downwash everywhere, then we would invert Eq. (6) as it stands to obtain an explicit solution for $p$, i.e.,

$$\frac{p}{\rho_\infty U_\infty^2} = \int_{-\infty}^{\infty} A(x-\xi) \frac{w(\xi)}{U_\infty} d\xi \tag{7}$$

where

$$p(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} p^* e^{i\alpha x} d\alpha$$
$$A(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i\alpha}{|\alpha|} e^{i\alpha x} d\alpha = -\frac{1}{\pi x} \tag{8}$$

This is the solution to the so-called aerodynamic "thickness" problem. However our interest here is in the "lifting" problem where we do not know $w$ off the wing but we do know that $p = 0$ there. Hence Eq. (6) is rewritten

$$w^*/U_\infty = (-i|\alpha|/\alpha) p^*/\rho_\infty U_\infty^2 \tag{9}$$

Inverting

$$\frac{w}{U_\infty} = \int_{\text{chord of wing}} K(x-\xi) \frac{p(\xi)}{\rho_\infty U_\infty^2} d\xi \tag{10}$$

where

$$K(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} -i \frac{|\alpha|}{\alpha} e^{i\alpha x} d\alpha = \frac{1}{\pi x} \tag{11}$$

(The similarity in form of $A$ and $K$ is fortuitous and does not usually extend to higher dimensions or the inclusion of compressibility or nonsteady effects.) Standard methods exist for solving integral equations such as Eq. (10). The present method has its usefulness in devising a means for determining the form of $K$.

*Three dimensions*

Defining a two-dimensional transform

$$p^* \equiv \iint_{-\infty}^{\infty} p(x, y, z) e^{-i\alpha x - i\gamma y} dx\, dy \tag{12}$$

we may solve the three-dimensional version of Eq. (1) subject to Eqs. (2–4). The results are